

EQUIVALENCES OF TRIANGULATED CATEGORIES AND FOURIER-MUKAI TRANSFORMS

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ABSTRACT. We give a condition for an exact functor between triangulated categories to be an equivalence. Applications to Fourier-Mukai transforms are discussed. In particular we obtain a large number of such transforms for K3 surfaces.

1. INTRODUCTION

Let X and Y be smooth projective varieties of the same dimension, and let \mathcal{P} be a vector bundle on $X \times Y$. Define a functor

$$F : D(Y) \longrightarrow D(X)$$

between the derived categories of sheaves on Y and X by the formula

$$F(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \otimes \pi_Y^*(-)),$$

where $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ are the projections maps. Functors of this type which are equivalences of categories are called *Fourier-Mukai transforms*, and have proved to be powerful tools for studying moduli spaces of vector bundles [4],[5],[11].

A vector bundle \mathcal{P} on $X \times Y$ is called *strongly simple* over Y if for each point $y \in Y$, the bundle \mathcal{P}_y on X is simple, and if for any two distinct points y_1, y_2 of Y , and any integer i , one has

$$\mathrm{Ext}_X^i(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0.$$

One might think of the family $\{\mathcal{P}_y : y \in Y\}$ as an ‘orthonormal’ set of bundles on X .

The following basic result allows one to construct many examples of Fourier-Mukai transforms.

Theorem 1.1. *The functor F is fully faithful if, and only if, \mathcal{P} is strongly simple over Y . It is an equivalence of categories precisely when one also has $\mathcal{P}_y = \mathcal{P}_y \otimes \omega_X$ for all $y \in Y$.*

The first statement is well-known [3],[8], but the second part is new. In this paper we shall prove Theorem 1.1, along with some more general results concerning exact functors between triangulated categories.

As an example of the use of Theorem 1.1, we have

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Corollary 1.2. *Let X be an algebraic K3 surface and let Y be a fine, compact, 2-dimensional moduli space of stable vector bundles on X . Then Y is also a K3 surface, and if \mathcal{P} is a universal bundle on $X \times Y$, the functor F is an equivalence of categories.*

Proof. The fact that Y is a K3 surface is Theorem 1.4 of [12]. Since ω_X is trivial, it is enough to check that \mathcal{P} is strongly simple over Y . This follows from [12], Proposition 3.12, because any stable sheaf which moves in a 2-dimensional moduli is semi-rigid. \square

Notation. All our schemes will be Noetherian schemes. A sheaf on a scheme X will mean a coherent \mathcal{O}_X -module, and a point of X will mean a closed point. If x is a point of X then \mathcal{O}_x denotes the structure sheaf of x with reduced scheme structure.

If a, b are objects of a triangulated category \mathcal{A} , put

$$\mathrm{Hom}_{\mathcal{A}}^i(a, b) = \mathrm{Hom}_{\mathcal{A}}(a, T^i b),$$

where $T : \mathcal{A} \rightarrow \mathcal{A}$ is the translation functor.

If X is a scheme, $D(X)$ will denote the bounded derived category of sheaves on X . For an object E of $D(X)$, let

$$E^\vee = \mathbf{R} \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(E, \mathcal{O}_X).$$

We shall write $H^i(E)$ for the i th cohomology sheaf of E , and $E[n]$ for the object obtained by shifting E to the left by n places. We say that E is a sheaf if $H^i(E) = 0$ when $i \neq 0$.

If $f : X \rightarrow Y$ is a morphism of schemes, and E is an object of $D(Y)$, $\mathbf{L}_p f^*(E)$ denotes the $(-p)$ th cohomology object of $\mathbf{L}f^*(E)$.

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2. FULLY FAITHFUL FUNCTORS

In this section we give a general criterion for an exact functor between triangulated categories to be fully faithful. Its proof is very similar to that of [14], Lemma 2.15.

Definition 2.1. Let \mathcal{A} be a triangulated category. A subclass Ω of the objects of \mathcal{A} will be called a *spanning class* for \mathcal{A} , if for any object a of \mathcal{A}

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}^i(\omega, a) = 0 \quad \forall \omega \in \Omega \quad \forall i \in \mathbb{Z} &\implies a \cong 0, \\ \mathrm{Hom}_{\mathcal{A}}^i(a, \omega) = 0 \quad \forall \omega \in \Omega \quad \forall i \in \mathbb{Z} &\implies a \cong 0. \end{aligned}$$

Example 2.2. If X is a smooth projective variety, then the set

$$\Omega = \{\mathcal{O}_x : x \in X\}$$

is a spanning class for $\mathcal{A} = D(X)$.

Proof. For any object a of $\mathcal{A} = D(X)$, and any $x \in X$, there is a spectral sequence

$$E_2^{p,q} = \text{Ext}_X^p(H^{-q}(a), \mathcal{O}_x) \implies \text{Hom}_{\mathcal{A}}^{p+q}(a, \mathcal{O}_x).$$

If a is non-zero, let q_0 be the maximal value of q such that $H^q(a)$ is non-zero, and assume that x is a closed point in the support of $H^{q_0}(a)$. Then there is a non-zero element of $E_2^{0,-q_0}$ which survives to give an element of $\text{Hom}_{\mathcal{A}}^{q_0}(a, \mathcal{O}_x)$. Serre duality then gives a non-zero element of $\text{Hom}_{\mathcal{A}}^i(\mathcal{O}_x, a)$, where $i = \dim X - q_0$. \square

Theorem 2.3. *Let \mathcal{A} and \mathcal{B} be triangulated categories and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor with a left and a right adjoint. Then F is fully faithful if, and only if, there exists a spanning class Ω for \mathcal{A} , such that for all elements ω_1, ω_2 of Ω , and all integers i , the homomorphism*

$$F : \text{Hom}_{\mathcal{A}}^i(\omega_1, \omega_2) \longrightarrow \text{Hom}_{\mathcal{A}}^i(F\omega_1, F\omega_2)$$

is an isomorphism.

Proof. One implication is clear, so let us assume the existence of Ω and prove that F is fully faithful.

Let $H : \mathcal{B} \longrightarrow \mathcal{A}$ be a right adjoint of F and let

$$\eta : 1_{\mathcal{A}} \longrightarrow H \circ F, \quad \epsilon : F \circ H \longrightarrow 1_{\mathcal{B}},$$

be the unit and counit respectively of the adjunction $F \dashv H$. Similarly, let $G : \mathcal{B} \longrightarrow \mathcal{A}$ be a left adjoint of F and let

$$\zeta : 1_{\mathcal{B}} \longrightarrow F \circ G, \quad \delta : G \circ F \longrightarrow 1_{\mathcal{A}},$$

be the unit and counit of $G \dashv F$. Note that by [14], Lemma 1.2, G and H are also exact functors.

For any pair of objects a and b of \mathcal{A} , and any integer i , there is a commutative diagram of group homomorphisms

$$(1) \quad \begin{array}{ccc} \text{Hom}_{\mathcal{A}}^i(a, b) & \xrightarrow{\eta(b)_*} & \text{Hom}_{\mathcal{A}}^i(a, HFb) \\ \delta(a)^* \downarrow & & \downarrow \beta \\ \text{Hom}_{\mathcal{A}}^i(GFa, b) & \xrightarrow{\alpha} & \text{Hom}_{\mathcal{B}}^i(Fa, Fb) \end{array}$$

in which $\alpha = \zeta(Fa)^* \circ F$ and $\beta = \epsilon(Fb)_* \circ F$ are isomorphisms, and the common diagonal is the map

$$F : \text{Hom}_{\mathcal{A}}^i(a, b) \longrightarrow \text{Hom}_{\mathcal{B}}^i(Fa, Fb).$$

When a and b are elements of Ω this map is an isomorphism (by hypothesis), so all the maps in (1) are isomorphisms.

First we show that for any object a in Ω , the morphism $\delta(a)$ is an isomorphism. To see this embed $\delta(a)$ in a triangle of \mathcal{A} :

$$GFa \xrightarrow{\delta(a)} a \longrightarrow c \longrightarrow T(GFa).$$

For any object b of Ω we can apply the functor $\mathrm{Hom}_{\mathcal{A}}(-, b)$ to this triangle and obtain a long exact sequence of groups

$$\begin{aligned} \cdots \longleftarrow \mathrm{Hom}_{\mathcal{A}}(GFa, b) \xleftarrow{\delta(a)^*} \mathrm{Hom}_{\mathcal{A}}(a, b) \longleftarrow \mathrm{Hom}_{\mathcal{A}}(c, b) \longleftarrow \\ \longleftarrow \mathrm{Hom}_{\mathcal{A}}^{-1}(GFa, b) \xleftarrow{\delta(a)^*} \mathrm{Hom}_{\mathcal{A}}^{-1}(a, b) \longleftarrow \cdots \end{aligned}$$

But since the maps $\delta(a)^*$ are all isomorphisms, this implies that

$$\mathrm{Hom}_{\mathcal{A}}^i(c, b) = 0 \quad \forall b \in \Omega \quad \forall i \in \mathbb{Z},$$

so $c \cong 0$ and $\delta(a)$ is an isomorphism.

Now take an object b of \mathcal{A} , embed the morphism $\eta(b)$ in a triangle

$$b \xrightarrow{\eta(b)} HFb \longrightarrow c \longrightarrow Tb,$$

and apply the functor $\mathrm{Hom}_{\mathcal{A}}(a, -)$ with $a \in \Omega$. The homomorphisms

$$\eta(a)_* : \mathrm{Hom}_{\mathcal{A}}^i(a, b) \longrightarrow \mathrm{Hom}_{\mathcal{A}}^i(a, HFb)$$

appearing in the resulting long exact sequence are isomorphisms because of the commuting diagram (1) and the fact that $\delta(a)^*$ is an isomorphism. Arguing as above we conclude that $c \cong 0$ and hence that $\eta(b)$ is an isomorphism. Since b was arbitrary, this is enough to show that F is fully faithful. \square

3. EQUIVALENCES OF TRIANGULATED CATEGORIES

Here we give a condition for a fully faithful exact functor between triangulated categories to be an equivalence. We refer to [9], VIII.2 for the notion of biproducts in an additive category.

Definition 3.1. A triangulated category \mathcal{A} will be called *indecomposable* if whenever \mathcal{A}_1 and \mathcal{A}_2 are full subcategories of \mathcal{A} satisfying

- (a) for every object a of \mathcal{A} there exist objects $a_j \in \mathrm{Ob}(\mathcal{A}_j)$ such that a is a biproduct of a_1 and a_2 ,
- (b) for any pair of objects $a_j \in \mathrm{Ob}(\mathcal{A}_j)$,

$$\mathrm{Hom}_{\mathcal{A}}^i(a_1, a_2) = \mathrm{Hom}_{\mathcal{A}}^i(a_2, a_1) = 0 \quad \forall i \in \mathbb{Z},$$

then there exists j such that $a \cong 0$ for all $a \in \mathrm{Ob}(\mathcal{A}_j)$.

Example 3.2. If X is a scheme then $D(X)$ is indecomposable if and only if X is connected.

Proof. We suppose that X is connected and prove that $\mathcal{A} = D(X)$ is indecomposable. The (easy) converse is left to the reader.

Suppose \mathcal{A}_1 and \mathcal{A}_2 are full subcategories of \mathcal{A} satisfying conditions (a) and (b) of the definition. For any integral closed subscheme Y of X , the sheaf \mathcal{O}_Y is indecomposable, and is therefore isomorphic to some object of \mathcal{A}_j , $j = 1$ or 2 . For any point $y \in Y$ we must then have that \mathcal{O}_y is also isomorphic to an object of \mathcal{A}_j , since otherwise (b) would imply that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{O}_Y, \mathcal{O}_y) = 0$, which is not the case.

Let X_j be the union of those Y such that \mathcal{O}_Y is isomorphic to an object of \mathcal{A}_j . Then X_1 and X_2 are closed subsets of X and $X = X_1 \cup X_2$. If a point $x \in X$ lies in X_1 and X_2 then \mathcal{O}_x is isomorphic to an object of \mathcal{A}_1 and to an object of \mathcal{A}_2 . This contradicts (b). Thus the union is disjoint, and the fact that X is connected implies that one of the X_j (without loss of generality X_2) is empty. But then (b) implies that for any object a of \mathcal{A}_2 one has

$$\mathrm{Hom}_{\mathcal{A}}^i(a, \mathcal{O}_x) = 0 \quad \forall i \in \mathbb{Z} \quad \forall x \in X,$$

and hence, by the argument of Example 2.2, $a \cong 0$. This completes the proof. \square

Theorem 3.3. *Let \mathcal{A} and \mathcal{B} be triangulated categories and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a fully faithful exact functor. Suppose that \mathcal{B} is indecomposable, and that not every object of \mathcal{A} is isomorphic to 0. Then F is an equivalence of categories if, and only if, F has a left adjoint G and a right adjoint H such that for any object b of \mathcal{B} ,*

$$Hb \cong 0 \implies Gb \cong 0.$$

Proof. If F is an equivalence then any quasi-inverse of F is a left and right adjoint for F . For the converse take an object b of \mathcal{B} and (with notation as in Theorem 2.3) embed the morphism $\epsilon(b)$ in a triangle of \mathcal{B} :

$$F H b \xrightarrow{\epsilon(b)} b \longrightarrow c \longrightarrow T(F H b).$$

Applying H one sees that $Hc \cong 0$, because the fact that F is fully faithful implies that the morphism $H(\epsilon(b))$ is an isomorphism. Define full subcategories \mathcal{B}_1 and \mathcal{B}_2 of \mathcal{B} consisting of objects satisfying $F H b \cong b$ and $Hb \cong 0$ respectively. Now our hypothesis implies that

$$\mathrm{Hom}_{\mathcal{B}}^i(b_1, b_2) = \mathrm{Hom}_{\mathcal{B}}^i(b_2, b_1) = 0 \quad \forall i \in \mathbb{Z},$$

whenever $b_j \in \mathcal{B}_j$. Furthermore, the lemma below applied to the triangle above shows that every object of \mathcal{B} is a biproduct $b_1 \oplus b_2$. Since \mathcal{B} is indecomposable we must have

$$Hc \cong 0 \implies c \cong 0,$$

for any $c \in \mathrm{Ob}(\mathcal{B})$, so the morphism $\epsilon(b)$ appearing above is an isomorphism. Since b was arbitrary, $F \circ H \cong 1_{\mathcal{B}}$ and F is an equivalence. \square

Lemma 3.4. *Let \mathcal{A} be a triangulated category and let*

$$a_1 \xrightarrow{i_1} b \xrightarrow{p_2} a_2 \xrightarrow{0} T a_1,$$

be a triangle of \mathcal{A} . Then b is a biproduct of a_1 and a_2 in \mathcal{A} .

Proof. Applying the functors $\mathrm{Hom}_{\mathcal{A}}(-, a_1)$ and $\mathrm{Hom}_{\mathcal{A}}(a_2, -)$, one obtains morphisms $p_1 : b \rightarrow a_1$ and $i_2 : a_2 \rightarrow b$, such that $p_1 \circ i_1 = 1_{a_1}$ and $p_2 \circ i_2 = 1_{a_2}$. The composition $p_2 \circ i_1$ is always 0 and replacing i_2

by $i_2 - i_1 \circ p_1 \circ i_2$, we can assume that $p_1 \circ i_2 = 0$. Then ([9], VIII.2) it is enough to check that the endomorphism of b given by

$$\phi = 1_b - i_1 \circ p_1 - i_2 \circ p_2$$

is the zero map. But this follows from the fact that $p_1 \circ \phi = p_2 \circ \phi = 0$. \square

4. INTEGRAL FUNCTORS

Throughout this section X and Y are smooth projective varieties over an algebraically closed field k of characteristic zero, and \mathcal{P} is an object of $D(X \times Y)$. F denotes the exact functor

$$\Phi_{Y \rightarrow X}^{\mathcal{P}} : D(Y) \longrightarrow D(X)$$

defined by the formula

$$\Phi_{Y \rightarrow X}^{\mathcal{P}}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \otimes^{\mathbf{L}} \pi_Y^*(-)).$$

Following Mukai, we call F an *integral functor*. Here we derive various general properties of such functors. Most of these appeared in some form in the original papers of Mukai on Abelian varieties [10],[11].

Given a scheme S , one can define a relative version of F over S . This is the functor

$$F_S : D(S \times Y) \longrightarrow D(S \times X),$$

given by the formula

$$F_S(-) = \mathbf{R}\pi_{S \times Y,*}(\mathcal{P}_S \otimes^{\mathbf{L}} \pi_{S \times X}^*(-)),$$

where $S \times X \xleftarrow{\pi_{S \times X}} S \times X \times Y \xrightarrow{\pi_{S \times Y}} S \times Y$ are the projection maps, and \mathcal{P}_S is the pull-back of \mathcal{P} to $S \times X \times Y$.

The following result is similar to [11], Proposition 1.3.

Lemma 4.1. *Let $g : T \rightarrow S$ be a morphism of schemes, and let E be an object of $D(S \times Y)$, of finite tor-dimension over S . Then there is an isomorphism*

$$F_T \circ \mathbf{L}(g \times 1_Y)^*(E) \cong \mathbf{L}(g \times 1_X)^* \circ F_S(E).$$

Proof. One needs to base-change around the diagram

$$\begin{array}{ccc} T \times X \times Y & \xrightarrow{(g \times 1_{X \times Y})} & S \times X \times Y \\ \pi_{T \times X} \downarrow & & \downarrow \pi_{S \times X} \\ T \times X & \xrightarrow{(g \times 1_X)} & S \times X \end{array}$$

This is justified by the same argument used to prove [3], Lemma 1.3. \square

We can now show that integral functors preserve families of sheaves. It is this property which makes them useful for studying moduli problems. See also [11], Theorem 1.6.

Proposition 4.2. *Let S be a scheme, and \mathcal{E} a sheaf on $S \times Y$, flat over S . Suppose that for each $s \in S$, $F(\mathcal{E}_s)$ is a sheaf on X . Then there is a sheaf $\hat{\mathcal{E}}$ on $S \times X$, flat over S , such that for every $s \in S$, $\hat{\mathcal{E}}_s = F(\mathcal{E}_s)$.*

Proof. Let $\hat{\mathcal{E}} = F_S(\mathcal{E})$, and take a point $s \in S$. Applying Lemma 4.1 with $T = \{s\}$, we see that the derived restriction of $\hat{\mathcal{E}}$ to the fibre $X \times \{s\}$ is just $F(\mathcal{E}_s)$. The following lemma then shows that $\hat{\mathcal{E}}$ is a sheaf on $S \times X$, flat over S . \square

Lemma 4.3. *Let $\pi : S \rightarrow T$ be a morphism of schemes, and for each point $t \in T$, let $i_t : S_t \rightarrow S$ denote the inclusion of the fibre $\pi^{-1}(t)$. Let \mathcal{E} be an object of $D(S)$, such that for all $t \in T$, $\mathbf{L}i_t^*(\mathcal{E})$ is a sheaf on S_t . Then \mathcal{E} is a sheaf on S , flat over T .*

Proof. For each point $t \in T$, consider the hypercohomology spectral sequence

$$E_2^{p,q} = \mathbf{L}_{-p}i_t^*(H^q(\mathcal{E})) \implies \mathbf{L}_{-(p+q)}i_t^*(\mathcal{E}).$$

By assumption, the right-hand side is zero unless $p + q = 0$. If q_0 is the largest q such that $H^q(\mathcal{E}) \neq 0$, then E_2^{0,q_0} survives in the spectral sequence for some $t \in T$, so $q_0 = 0$. Now $H^0(\mathcal{E})$ must be flat over T since otherwise $E_2^{-1,0}$ would survive for some $t \in T$. Finally, suppose $H^q(\mathcal{E}) \neq 0$ for some $q < 0$. Then we can find a largest such q , and this gives an element of $E_2^{0,q}$ which survives. Hence $H^q(\mathcal{E}) = 0$ unless $q = 0$ and \mathcal{E} is a sheaf, flat over T . \square

In the next section we shall need

Lemma 4.4. *Suppose that \mathcal{P} is a sheaf on $X \times Y$, flat over Y , and fix a point $y \in Y$. Then the homomorphism*

$$(2) \quad F : \mathrm{Ext}_Y^1(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Ext}_X^1(\mathcal{P}_y, \mathcal{P}_y),$$

is the Kodaira-Spencer map for the family \mathcal{P} at the point y , if we identify the first space with the tangent space to Y at y in the usual way.

Proof. Let $D = \mathrm{Spec} k[\epsilon]/\epsilon^2$ denote the double point. We identify the tangent space $T_y Y$ to Y at y with the set of morphisms $D \rightarrow Y$, taking the closed point of D to y . Given such a morphism f , we can pull \mathcal{P} back, and obtain a deformation of the sheaf \mathcal{P}_y on X , with base D . The set of such deformations is identified with $\mathrm{Ext}_X^1(\mathcal{P}_y, \mathcal{P}_y)$, and the Kodaira-Spencer map is the resulting linear map

$$T_y Y \longrightarrow \mathrm{Ext}_X^1(\mathcal{P}_y, \mathcal{P}_y).$$

Returning to our homomorphism (2), note that we can identify the domain with the set of deformations of \mathcal{O}_y over D , and the image with the set of deformations of \mathcal{P}_y over D . If we do this, it is easy to see that the map F is just given by applying the functor F_D .

Given an element $f : D \rightarrow Y$ of $T_y Y$, the corresponding deformation of \mathcal{O}_y over D is obtained by pulling-back the family \mathcal{O}_Δ on $Y \times Y$ to $D \times Y$ using f (here Δ denotes the diagonal in $Y \times Y$). By Lemma 4.1, if we then apply F_D , we get the same result as if we first applied F_Y , which gives the sheaf \mathcal{P} on $X \times Y$, and then pulled-back via f . But this is the Kodaira-Spencer map for the family \mathcal{P} . \square

The following result is well-known. Its proof is a straightforward application of Grothendieck-Verdier duality (see [8], Proposition 3.1 or [3], Lemma 1.2).

Lemma 4.5. *The functors*

$$G = \Phi_{X \rightarrow Y}^{\mathcal{P}^\vee \otimes \pi_X^* \omega_X [\dim X]}, \quad H = \Phi_{X \rightarrow Y}^{\mathcal{P}^\vee \otimes \pi_Y^* \omega_Y [\dim Y]},$$

are left and right adjoints for F respectively. \square

5. APPLICATIONS TO FOURIER-MUKAI TRANSFORMS

As in the last section we fix smooth projective varieties X and Y over an algebraically closed field k of characteristic zero, and an object \mathcal{P} of $D(X \times Y)$. F denotes the corresponding functor $\Phi_{Y \rightarrow X}^{\mathcal{P}}$. The following theorem was first proved by A.I. Bondal and D.O. Orlov, using ideas of Mukai.

Theorem 5.1. ([3]) *The functor F is fully faithful if, and only if, for each point $y \in Y$,*

$$\mathrm{Hom}_{D(X)}(F\mathcal{O}_y, F\mathcal{O}_y) = k,$$

and for each pair of points $y_1, y_2 \in Y$, and each integer i ,

$$\mathrm{Hom}_{D(X)}^i(F\mathcal{O}_{y_1}, F\mathcal{O}_{y_2}) = 0 \text{ unless } y_1 = y_2 \text{ and } 0 \leq i \leq \dim Y.$$

Proof. We must show that for any point y of Y , and any integer i , the homomorphism

$$F : \mathrm{Hom}_{D(Y)}^i(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D(X)}^i(F\mathcal{O}_y, F\mathcal{O}_y)$$

is an isomorphism. Theorem 2.3 will then give the result. By the commutative diagram (1) it will be enough to show that $\delta(\mathcal{O}_y)$ is an isomorphism. In fact it will be enough to show that $GF\mathcal{O}_y \cong \mathcal{O}_y$, because then $\delta(\mathcal{O}_y)$ must be either an isomorphism or zero, and the latter is impossible, because $F(\delta(\mathcal{O}_y))$ has a left-inverse.

For any point $z \xrightarrow{i_z} Y$, there are isomorphisms of vector spaces

$$\mathbf{L}_p i_z^*(GF\mathcal{O}_y) \cong \mathrm{Hom}_{D(Y)}^p(GF\mathcal{O}_y, \mathcal{O}_z) \cong \mathrm{Hom}_{D(X)}^p(F\mathcal{O}_y, F\mathcal{O}_z)$$

coming from the adjunctions $i_z^* \dashv i_{z,*}$ ([7], Corollary 5.11), and $G \dashv F$. Thus, by [3], Proposition 1.5, $GF\mathcal{O}_y$ is a sheaf supported at the point y . Furthermore, there is a unique morphism $GF\mathcal{O}_y \rightarrow \mathcal{O}_y$. If K is the kernel of this morphism, one has a short exact sequence

$$0 \longrightarrow K \longrightarrow GF\mathcal{O}_y \xrightarrow{\delta(\mathcal{O}_y)} \mathcal{O}_y \longrightarrow 0,$$

and we must show that $K = 0$. Applying the functor $\mathrm{Hom}_{D(Y)}(-, \mathcal{O}_y)$, and using the diagram (1), it will be enough to show that the homomorphism

$$(3) \quad F : \mathrm{Hom}_{D(Y)}^1(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D(X)}^1(F\mathcal{O}_y, F\mathcal{O}_y),$$

is injective.

By [10], Proposition 1.3, $GF = \Phi_{Y \rightarrow Y}^{\mathcal{Q}}$ for some object \mathcal{Q} of $D(Y \times Y)$. Since $GF\mathcal{O}_y$ is a sheaf for all $y \in Y$, Lemma 4.3 shows that \mathcal{Q} is in fact a sheaf, flat over Y . Furthermore, by Lemma 4.4, the map

$$GF : \mathrm{Hom}_{D(Y)}^1(\mathcal{O}_y, \mathcal{O}_y) \longrightarrow \mathrm{Hom}_{D(Y)}^1(GF\mathcal{O}_y, GF\mathcal{O}_y),$$

is given by the Kodaira-Spencer map for the family \mathcal{Q} at the point y . The following two lemmas show that this map is injective. Clearly the map (3) must also be injective. \square

Lemma 5.2. *Let Y be a projective variety over k , and let Q be a sheaf on Y supported at a point $y \in Y$. Suppose that*

$$\mathrm{Hom}_Y(Q, \mathcal{O}_y) = k.$$

Then Q is the structure sheaf of a zero-dimensional closed subscheme of Y .

Proof. There exists a short exact sequence

$$0 \longrightarrow P \longrightarrow Q \xrightarrow{g} \mathcal{O}_y \longrightarrow 0.$$

Suppose $f : \mathcal{O}_Y \rightarrow Q$ is a non-surjective morphism of sheaves. Considering the cokernel of f shows that there is a non-zero morphism $h : Q \rightarrow \mathcal{O}_y$ such that $h \circ f = 0$. But by hypothesis h must be a multiple of g , so one must have $g \circ f = 0$, hence f comes from a morphism $\mathcal{O}_Y \rightarrow P$. Now

$$\dim_k H^0(Y, P) = \chi(P) < \chi(Q) = \dim_k H^0(Y, Q),$$

so there must be a morphism $\mathcal{O}_Y \rightarrow Q$ which is surjective. \square

Lemma 5.3. *Let S and Y be varieties over k , with Y projective. Let \mathcal{Q} be a sheaf on $S \times Y$, flat over S , such that for each $s \in S$, \mathcal{Q}_s is the structure sheaf of a zero-dimensional closed subscheme of Y . Suppose also that for all pairs of points $s_1, s_2 \in S$*

$$(4) \quad \mathcal{Q}_{s_1} \cong \mathcal{Q}_{s_2} \implies s_1 = s_2.$$

Then there exists a point $s \in S$, such that the Kodaira-Spencer map for the family \mathcal{Q} at s is injective.

Proof. Firstly, we may suppose that S is affine. Fix a point $s \in S$, and let $\pi : S \times Y \rightarrow S$ be the projection map. By the theorem on cohomology and base-change, the natural map

$$H^0(S \times Y, \mathcal{Q}) \rightarrow H^0(Y, \mathcal{Q}_s)$$

is surjective, so we can find a section $g : \mathcal{O}_{S \times Y} \rightarrow \mathcal{Q}$ such that the restriction $g_s : \mathcal{O}_Y \rightarrow \mathcal{Q}_s$ is surjective. Passing to an open subset of S we can assume that g is surjective, so that \mathcal{Q} is the structure sheaf of a closed subscheme of $S \times Y$.

Let P be the (constant) Hilbert polynomial of the sheaf \mathcal{Q}_s on Y . By the general existence theorem for Hilbert schemes [6], there is a scheme $\text{Hilb}^P(Y)$ representing the functor which assigns to a scheme S the set of S -flat quotients \mathcal{Q} of $\mathcal{O}_{S \times Y}$ with Hilbert polynomial P . Let \mathcal{E} be the universal quotient on $\text{Hilb}^P(Y) \times Y$. Then there is a morphism $f : S \rightarrow \text{Hilb}^P(Y)$ such that $\mathcal{Q} = (f \times 1_Y)^*(\mathcal{E})$. The Kodaira-Spencer map for the family \mathcal{Q} at $s \in S$ is obtained by composing the Kodaira-Spencer map for the family \mathcal{E} at $f(s)$ with the differential

$$T_s(f) : T_s S \longrightarrow T_{f(s)} \text{Hilb}^P(Y).$$

Now condition (4) implies that the morphism f is injective on points. Let S' be its scheme-theoretic image. Since we are in characteristic zero we can assume S, S' are non-singular, and $f : S \rightarrow S'$ is smooth. This implies that for some $s \in S$, $T_s(f)$ is injective. Finally, the fact that the Kodaira-Spencer map for the family \mathcal{E} is injective is a consequence of the universal property of \mathcal{E} . This completes the proof. \square

Theorem 3.3 allows us to say when F is an equivalence.

Theorem 5.4. *Suppose F is fully faithful. Then F is an equivalence if, and only if, for every point $y \in Y$,*

$$(5) \quad F\mathcal{O}_y \otimes \omega_X \cong F\mathcal{O}_y.$$

Proof. Let G and H denote the left and right adjoint functors of F respectively. Suppose first that F is an equivalence. Then G and H are both quasi-inverses for F , so for any $y \in Y$,

$$G(F\mathcal{O}_y) \cong H(F\mathcal{O}_y) \cong \mathcal{O}_y.$$

From the formulas for G and H given in Lemma 4.5,

$$G(F\mathcal{O}_y) \cong G(F\mathcal{O}_y) \otimes \omega_Y \cong H(F\mathcal{O}_y \otimes \omega_X)[\dim X - \dim Y].$$

But G is an equivalence, so one concludes that X and Y have the same dimension, and there is an isomorphism (5).

For the converse, let X have dimension n , and suppose that (5) holds for all $y \in Y$. Take an object b of $D(X)$ such that $Hb \cong 0$. For any point $y \in Y$, and any integer i ,

$$\begin{aligned} \text{Hom}_{D(Y)}^i(Gb, \mathcal{O}_y) &= \text{Hom}_{D(X)}^i(b, F\mathcal{O}_y) = \text{Hom}_{D(X)}^i(b, F\mathcal{O}_y \otimes \omega_X) \\ &= \text{Hom}_{D(X)}^{n-i}(F\mathcal{O}_y, b)^\vee = \text{Hom}_{D(Y)}^{n-i}(\mathcal{O}_y, Hb)^\vee = 0, \end{aligned}$$

so by Example 2.2, $Gb \cong 0$. Applying Theorem 3.3 completes the proof. \square

Finally note that Theorems 5.1 and 5.4 imply Theorem 1.1 in the special case when \mathcal{P} is a vector bundle on $X \times Y$.

REFERENCES

- [1] *C. Bartocci, U. Bruzzo, D. Hernandez Ruiperez*, A Fourier-Mukai transform for stable bundles on K3 surfaces, *J. reine angew. Math.* **486** (1997), 1-16.
- [2] *A.I. Bondal*, Representations of associative algebras and coherent sheaves, *Math. USSR Izv.* **34** (1990), 23-42.
- [3] *A.I. Bondal, D.O. Orlov*, Semiorthogonal decomposition for algebraic varieties, Preprint alg-geom/9506012.
- [4] *T. Bridgeland*, Fourier-Mukai transforms for elliptic surfaces, Preprint alg-geom/9705002, to appear in *J. reine angew. Math.*
- [5] *U. Bruzzo, A. Maciocia*, Hilbert schemes of points on some K3 surfaces and Gieseker stable bundles, *Math. Proc. Cam. Phil. Soc.* **120** (1996), 255-261.
- [6] *A. Grothendieck*, Techniques de construction et théorèmes d'existence en géométrie algébrique, IV : les schémas de Hilbert, *Sem. Bourbaki* **221** (1960).
- [7] *R. Hartshorne*, Residues and duality, *Lect. Notes Math.* **20**, Springer-Verlag, Heidelberg (1966).
- [8] *A. Maciocia*, Generalized Fourier-Mukai transforms, *J. reine angew. Math.* **480** (1996), 197-211.
- [9] *S. MacLane*, Categories for the working mathematician, *Grad. Texts Math.* **5**, Springer-Verlag, Berlin (1971).
- [10] *S. Mukai*, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* **81** (1981), 153-175.
- [11] *S. Mukai*, Fourier functor and its application to the moduli of bundles on an abelian variety, *Adv. Pure Math.* **10** (1987), 515-550.
- [12] *S. Mukai*, On the moduli space of bundles on K3 surfaces I, in: *Vector Bundles on Algebraic Varieties*, M.F. Atiyah et al., Oxford University Press (1987), 341-413.
- [13] *S. Mukai*, Duality of polarized K3 surfaces, to appear in *Proc. of Algebraic Geometry Euroconference (Warwick 1996)*.
- [14] *D.O. Orlov*, Equivalences of derived categories and K3 surfaces, Preprint alg-geom/9606006.

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